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A Regularized Approach to Universal Orbit Variables

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Nomenclature

$\hat{C}$ = universal variable, Eq. (13)
D = $r\dot{r}/\nu$
E, F = eccentric anomaly, elliptic and hyperbolic
$\mathfrak{F}, \mathfrak{G}$ = representation coefficients, Eqs. (24) and (30)
$\hat{M}$ = universal time variable, Eq. (14)
$\mathbf{R}$ = $\mathbf{r}\,r^{-1/2}$
$\hat{S}, \hat{U}$ = universal variables, Eqs. (13) and (15)
T = time of pericenter passage
$\hat{X}$ = universal eccentric anomaly, Eq. (4)
a = semimajor axis
e = eccentricity and base of natural logarithms
f, g = representation coefficients, Eq. (26)
h = Eq. (34)
i = orbit inclination angle
$\mathbf{r}, r$ = position vector, radius
$\dot{\mathbf{r}}, \dot{r}$ = velocity and radial component
$\mathbf{r}', r'$ = transformed velocity and radial component
v = true anomaly
α = energy constant, $-1/a$
β = $(\alpha)^{1/2}\hat{X}$
ν = gravitational constant, Eq. (1)
τ = time variable
ω = argument of pericenter
Ω = longitude of ascending node

Superscripts and subscripts

$(\dot{\ })$ = $d(\ )/d\tau$
$(\ )'$ = $d(\ )d\hat{X}$
0 = evaluated at epoch τ_0

I. Introduction

The recent introduction of rocket propulsors for space flight has presented possibilities of encountering orbits whose character changes significantly during the time of interest. For instance, a low-acceleration escape trajectory might begin in a nearly circular orbit and terminate on a hyperbolic path. This aspect, as well as the nearly universal use today of electronic computers for orbit computation and the attendant desire to simplify programing, motivate one to employ a formulation free from possibilities of indeterminateness, no matter what geometric form the orbit takes, be it elliptic, parabolic, hyperbolic, or possibly a circular or rectilinear limit.

In 1947 Stumpff[1] found a formulation free of indeterminateness by algebraic rearrangement of the classical orbit formulas. Herrick[2] more recently introduced a related but algebraically simpler formulation, again by an heuristic rearrangement of classical forms so that elliptic, parabolic, and hyperbolic cases could be expressed in terms of a generalized eccentric anomaly $\hat{X}$, e.g., $\hat{X} = (a)^{1/2}(E - E_0)$ in elliptic orbits. In both methods the classical elements a, e, i, Ω, ω, and T are replaced by the six components of position and velocity at some reference epoch $\mathbf{r}_0 = \mathbf{r}(\tau_0)$, $\dot{\mathbf{r}}_0 = \dot{\mathbf{r}}(\tau_0)$. These vectors have significance in a two-body orbit of any geometric shape and, in themselves, introduce no indeterminateness, as contrasted to the Euler angles i, Ω, ω.

The purpose here is to derive Herrick's universal expressions via regularization and integration of the two-body differential equations of motion and thereby place the universal variables on a somewhat more formal mathematical foundation. As a result of this approach, an alternate set of "conditioned" elements, which remain bounded even at the origin arises naturally. Because the universal formulation is developed primarily for numerical work with electronic computers, the proper computational sequences so important to maintain good accuracy and high speed also will be discussed.

II. Regularizing Transformation

The vector differential equation for the relative motion of a pair of mutually attracting mass points with inverse square gravitational fields is

$$\ddot{\mathbf{r}} = -\nu^2\mathbf{r}/r^3 \tag{1}$$

where ν is the square root of the usual gravitational constant μ. A first integral, the energy or Via Viva, is obtained after taking the inner product with $\dot{\mathbf{r}}$

$$\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}/\nu^2 = 2/r + \alpha \tag{2}$$

where α, the integration constant, may be determined by this equation evaluated at a particular epoch τ_0.

Inspection of the equation of motion (1) reveals a second-order singularity at $r = 0$. A solution valid in the neighborhood of this singularity will be possible only if the point can be regularized by a change of variables so that the singularity becomes a regular point in the transformed coordinates. The regularizing transformation that will be used here is that introduced by Sundman[3] in connection with the three-body problem.† It may be expressed as

$$d(\ )/d\hat{X} = (r/\nu)\, d(\ )/d\tau \tag{3}$$

or in the integrated form

$$\hat{M} \triangleq \nu(\tau - \tau_0) = \int_0^{\hat{X}} r d\hat{X} \tag{4}$$

where $\hat{X}$ is taken to be zero at τ_0.

Application of (3) to (1) then gives the transformed equation of motion regular at $r = 0$ where the prime denotes differentiation with respect to $\hat{X}$

$$\mathbf{r}'' - r'\mathbf{r}'/r + \mathbf{r}/r = 0 \tag{5}$$

The magnitude of the third term is unity, so it gives no difficulty. To show that the second term also remains

Received December 15, 1964; revision received April 5, 1965. Condensed from portions of the author's Ph.D. Dissertation.[4] This work was supported partially by the NASA Predoctoral Trainee program.

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† Sperling[6] also has used the Sundman transformation and the energy integral to obtain a different form of Eq. (5), whose solution leads to a set of related universal variables that are functions of β only, but which also require carrying powers of $\hat{X}$ separately. Battin[7] employs a still different version (also functions of β only). Both sets can be deduced from the variables treated here by carrying the first terms of Eqs. (32) separately, plus some algebraic rearrangement. The author wishes to express his thanks to the reviewer for calling his attention to Ref. 6

bounded near the origin, apply the transformation to the energy integral (2) and obtain

$$\mathbf{r}'\cdot\mathbf{r}' = 2r + \alpha r^2 \tag{6}$$

Noting that r' is the radial component of $\mathbf{r}'$, it follows that

$$|r'\mathbf{r}'/r| \leq |\mathbf{r}'\cdot\mathbf{r}'/r| = |2 + \alpha r| \tag{7}$$

which is bounded when $r \to 0$. Equation (6) also shows that the transformed speed behaves as $(r)^{1/2}$ near the origin, whereas the real speed from Eq. (2) behaves as $(r)^{-1/2}$.

It easily can be proved[4] that the only circumstance under which r can become arbitrarily small is the case of a rectilinear orbit. In this rectilinear case, the equality holds in (7), and the equations of motion reduce to the simple linear equation

$$r'' = 1 + \alpha r \tag{8}$$

Interestingly enough, this scalar equation for the radius also holds for any two-body orbit shape. Since $r^2 = \mathbf{r}\cdot\mathbf{r}$, one has

$$\tfrac{1}{2}[d^2(r^2)/d\tau^2] = \dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + \mathbf{r}\cdot\ddot{\mathbf{r}} = \nu^2[(1/r) + \alpha] \tag{9}$$

the latter being obtained upon substitution of (1) and (2). Noting that the Sundman transformation gives

$$r' = r\,\dot{r}/\nu \;(= D) \tag{10}$$

Eq. (9) becomes

$$\nu^{-1}dr'/d\tau = 1/r + \alpha \tag{11}$$

which, upon multiplication by r and another application of the transformation (3), yields (8).

III. Integration of Equations of Motion

Integration of Eq. (8) with initial conditions $r_0 = (r_0'' - 1)/\alpha$, r_0', and with α, evaluated from (5) at τ_0, gives

$$r = r_0 + r_0'\left[\frac{e^\beta - e^{-\beta}}{2(\alpha)^{1/2}}\right] + r_0''\left(\frac{e^\beta + e^{-\beta} - 2}{2\alpha}\right) \tag{12}$$

where $\beta = (\alpha)^{1/2}\hat{X}$. Letting the terms within parentheses be designated by $\hat{S}(\alpha, \hat{X})$ and $\hat{C}(\alpha, \hat{X})$, respectively, to define two "universal variables" gives

$$r = r_0 + r_0'\hat{S} + r_0''\hat{C} \tag{13}$$

This gives r only as a function of $\hat{X}$, so it is necessary to relate $\hat{X}$ to time. The integrated form of the Sundman transformation (4) serves this purpose, giving

$$\hat{M} \triangleq \nu(\tau - \tau_0) = r_0\hat{X} + r_0'\hat{C} + \hat{U}r_0'' \tag{14}$$

a transcendental equation in $\hat{X}$, which will be called the "universal Kepler equation." The additional universal variable appearing here must be

$$\hat{U}(\alpha, \hat{X}) \triangleq \int_0^{\hat{X}} \hat{C}d\hat{X} = \frac{e^\beta - e^{-\beta}}{2(\alpha)^{3/2}} - \frac{\hat{X}}{\alpha} \tag{15}$$

The following relations now may be verified from the preceding definitions.

$$\hat{S} = \hat{X} + \alpha\hat{U} \tag{16}$$

$$\hat{S}^2 = 2\hat{C} + \hat{C}^2 \tag{17}$$

$$\hat{U}' = \hat{C} \tag{18}$$

$$\hat{C}' = \hat{S} \tag{19}$$

$$\hat{S}' = 1 + \alpha\hat{C} \tag{20}$$

It is apparent that (13) and (14) give not only a solution for the radius in general but also a complete solution for the rectilinear case. Since the solution of the nonlinear vector equation for $\mathbf{r}$ must be consistent in the limit with the rectilinear case, a vector analogy to (14) is suggested

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{r}_0'\hat{S} + \mathbf{r}_0''\hat{C} \tag{21}$$

Table 1 Relations between classical and universal variables for ellipse, parabola, and hyperbola

	Ellipse	Parabola	Hyperbola
α	$= -1/a$	0	$1/a$
$\hat{S}$	$= (a)^{1/2}(E - E_0)$	$D - D_0$	$(-a)^{1/2}(F - F_0)$
$\hat{X}$	$= (a)^{1/2}\sin(E - E_0)$	$D - D_0$	$(-a)^{1/2}\sinh(F - F_0)$
$\hat{C}$	$= a[1 - \cos(E - E_0)]$	$\frac{1}{2}(D = D_0)^2$	$a[1 - \cosh(F - F_0)]$
$\hat{U}$	$= a^{3/2}[(E - E_0) - \sin(E - E_0)]$	$\frac{1}{6}(D - D_0)^3$	$(-a)^{3/2}[\sinh(F - F_0) - (F - F_0)]$
$\hat{M}$	$= a^{3/2}(M - M_0)$	$M - M_0$	$(-a)^{3/2}(M - M_0)$

Upon substitution into the original differential equation [Eq. (5)] and taking uniqueness in the absence of singularities into account, this is verified as the general solution with the aid of (17). Note that here one has $\mathbf{r}_0'' = \mathbf{r}_0''(\mathbf{r}_0, \mathbf{r}_0')$ from (5), so only six independent constants of integration appear as required. This relation in conjunction with the solution of the generalized Kepler equation for $\hat{X}(\tau)$ and the definitions of the universal variables $\hat{U}$, $\hat{C}$, and $\hat{S}$ then comprise a complete solution for $\mathbf{r}(\tau)$.

The transformed velocity is obtained by differentiating (21)

$$\mathbf{r}' = \mathbf{r}_0'(1 + \alpha\hat{C}) + \mathbf{r}_0''\hat{S} \tag{22}$$

Eliminating $\mathbf{r}_0''$ with the equation of motion then gives

$$\begin{aligned} r &= (1 - \hat{C}/r_0)\mathbf{r}_0 + (\hat{S} + r_0'\hat{C}/r_0)\mathbf{r}_0' \\ r' &= -(\hat{S}/r_0)\mathbf{r}_0 + (1 + \alpha\hat{C} + r_0'\hat{S}/r_0)\mathbf{r}_0' \end{aligned} \tag{23}$$

or

$$\mathbf{r} = f\mathbf{r}_0 + \mathcal{G}\mathbf{r}_0' \qquad \mathbf{r}' = f'\mathbf{r}_0 + \mathcal{G}'\mathbf{r}_0' \tag{24}$$

where the coefficients f, $\mathcal{G}$, f', and $\mathcal{G}'$ are called "representation coefficients" in the transformed coordinates.

Upon applying the inverse transformation to $\mathbf{r}'$ and $\mathbf{r}_0'$ and simplification with (13, 14, and 16), (24) becomes

$$\mathbf{r}(\tau) = f(\tau)\mathbf{r}_0 + g(\tau)\dot{\mathbf{r}}_0 \qquad \dot{\mathbf{r}}(\tau) = \dot{f}(\tau)\mathbf{r}_0 + \dot{g}(\tau)\dot{\mathbf{r}}_0 \tag{25}$$

which is merely an expression of the fact that the motion is coplanar, and that any vector in the plane of motion is a linear combination of two independent vectors $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$. This form is common to orbit determination theory. The time-varying "representation coefficients" usually are expressed in a rearranged Taylor series expansion in time, the classical "f and g series" introduced by Lagrange, which are convergent and computationally useful for small enough time intervals, thus providing a locally universal solution. Here, however, closed-form universal expressions have been obtained for the representation coefficients

$$\begin{aligned} f &= 1 - \hat{C}/r_0 \qquad & g &= r_0\mathcal{G}/\nu = (\hat{M} - \hat{U})/\nu \\ \dot{f} &= \nu f'/r = -\nu\hat{S}/rr_0 \qquad & \dot{g} &= r_0\mathcal{G}'/r = 1 - \hat{C}/r \end{aligned} \tag{26}$$

Equations (25) with coefficients (26), in terms of $\hat{X}$ obtained from solution of (14) and r from (13), constitute a universal solution of the two-body problem. Note that the only possibility of a small divisor that can arise in this formulation is when $r \to 0$, a rather unlikely occurrence in most real orbits. The relationship of the universal variables to the classical forms in terms of eccentric and true anomalies is given in Table 1, taken from Ref. 2. It should be noted here that in the parabolic case

$$D - D_0 = p^{1/2}[\tan(v/2) - \tan(v_0/2)] \tag{27}$$

IV. Conditioned Elements

Representation in terms of $\mathbf{r}_0$, $\dot{\mathbf{r}}_0$, and the representation coefficients (26) may lead to possible numerical difficulty within the region near the origin where f and g become large because of the factor r in their denominators. This, of course, reflects the fact that the speed becomes infinite at the origin. A numerical problem also may arise when integrating these elements to account for effects of perturbations, since the magnitude of the epoch velocity vector $|\dot{\mathbf{r}}_0|$ varies inversely with that of the position vector $|\mathbf{r}_0|$, e.g., the case of a sustained low-thrust escape trajectory. This "ill conditioning" can contribute to increased cross-feeding of errors from large to small components and also limit integration step size, thus leading to inefficiency.

These difficulties can, to some degree, be avoided by retaining the transformed coefficients and elements of Eq. (23) through most of the computations with only a single division of $\mathbf{r}'$ by r/ν remaining to obtain $\dot{\mathbf{r}}$ in cases where the actual velocity is desired. A further step toward better conditioning for integration purposes is to replace $\mathbf{r}_0$ by

$$\mathbf{R}_0 = \mathbf{r}_0/(r_0)^{1/2} \tag{28}$$

This new "position" vector then varies in magnitude at nearly the same rate as the transformed speed under perturbing influences (they are identical in circular orbits). The representation formulas then become

$$\mathbf{r} = \mathfrak{F}\mathbf{R}_0 + \mathfrak{G}\mathbf{r}_0' \qquad \mathbf{r}' = \mathfrak{F}'\mathbf{R}_0 + \mathfrak{G}'\mathbf{r}_0' \tag{29}$$

where it follows that

$$\mathfrak{F} = (r_0)^{1/2}f \qquad \mathfrak{F}' = (r_0)^{1/2}f' \tag{30}$$

V. Computational Aspects

In a typical representation problem such as ephemeris generation or in differential correction, one starts with a set of elements $\mathbf{r}_0$, $\dot{\mathbf{r}}_0$ or $\mathbf{R}_0$, $\mathbf{r}_0'$ and computes position and velocity at a later time. This involves solution of the universal Kepler equation [Eq. (14)] for $\hat{X}$ and simultaneous evaluation of the universal variables $\hat{U}$, $\hat{C}$, and $\hat{S}$ with high precision. The procedure will be outlined here.

The Newton-Raphson technique for solution of (14) will be most convenient. First make an initial estimate of $\hat{X}$. If the time interval since the last representation is not too large, the simple form

$$\Delta\hat{X} = \dot{\hat{X}}\Delta\tau = \nu\Delta\tau/r \tag{31}$$

will suffice to increment $\hat{X}$ from its previous value. For larger time intervals or an initial estimate, a systematic search may be necessary.[4]

A provisional computation of the universal variables must be made now. The exponential forms (12) and (15) are not suitable since they involve numerical indeterminateness at $\alpha = 0$ (the parabolic case). Further, one should avoid expressing these variables in terms of the circular or hyperbolic functions given in Table 1 since this would destroy the universality of the formulation. The series expansions, on the other hand, remain determinate and unique for all of the orbit shapes. Thus, one has for computation

$$\left.\begin{aligned} \hat{U} &= \frac{\hat{X}^3}{3!}\left\{1 + \frac{\beta^2}{4.5}\left[1 + \frac{\beta^2}{6.7}(1 + \ldots)\right]\right\} \\ \hat{C} &= \frac{\hat{X}^2}{2!}\left\{1 + \frac{\beta^2}{3.4}\left[1 + \frac{\beta^2}{5.6}(1 + \ldots)\right]\right\} \\ \hat{S} &= \hat{X} + \alpha\hat{U} \end{aligned}\right\} \tag{32}$$

The factored forms are ideal to minimize roundoff error. By noting that $\hat{C}$ and $\hat{S}$ are periodic in β if α is negative (or by reference to Table 1), it may be verified also that, in terms of the eccentric anomaly,

$$\beta^2 = \alpha\hat{X}^2 = (E - E_0)^2 \tag{33}$$

so that, if the epoch value E_0 is shifted by -2π every revolution at the point where $E - E_0 = \pi$, the magnitude of β^2 then will be limited to $\leq\pi^2$. In this event, only eight terms of the series will be required to maintain eight significant-figure accuracy in the computation of the universal variables. For hyperbolic orbits where β^2 cannot be limited easily, the number of terms required can be quickly determined, as shown in Ref. 4. For instance, only sixteen terms will be required for $\beta^2 = 100$. Note that β also tends to remain small for nearly parabolic orbits ($\alpha \to 0$), and the series automatically truncate themselves at the first term at the parabolic limit.

The shifting operation for elliptic orbits is reflected in both the $\hat{X}$ and $\hat{M}$ variables. A test is made first to ascertain that $\alpha < 0$ and then to see if $\hat{M}(-\alpha)^{3/2} \geq \pi$. If both conditions are met, $\hat{M}$ is changed by $-2\pi(-\alpha)^{-3/2}$ and $\hat{X}$ by $-2\pi(-\alpha)^{-1/2}$. In the case of small negative value of α, double precision arithmetic can be employed for the shifting operations to preserve accuracy in $\hat{M}$, the time variable.

With provisional values of the variables, the iterative solution of the generalized Kepler equation can proceed. Designating

$$h(\hat{X}) \triangleq \hat{M} - r_0\hat{X} - r'\hat{C} - r_0''\hat{U} \tag{34}$$

it will be noted that

$$h'(\hat{X}) = -r \tag{35}$$

with r computed from (13), so the correction to $\hat{X}$ becomes

$$\delta\hat{X} = h/r \tag{36}$$

The universal variables $\hat{U}$, $\hat{C}$, and $\hat{S}$ then are recomputed from (32) and the correction repeated until the iteration closes. The obvious temptation to use first-order differential correction formulas that arise from the recursion formulas [Eqs. (18–20)], i.e.,

$$\delta\hat{U} \cong \hat{C}\delta\hat{X} \qquad \delta\hat{C} \cong \hat{S}\delta\hat{X} \tag{37}$$

must be avoided since the universal Kepler equation then can be satisfied with erroneous values of $\hat{C}$ and $\hat{U}$. It is shown in Ref. 4 that Taylor series expansions in $\delta\hat{X}$, requiring as many terms as the original series (32), are in fact required to maintain acceptable accuracy.

The iteration may be terminated when $|\delta\hat{X}| < \epsilon$, a small predetermined quantity. A convenient method is to employ Ostrowski's[5] existence theorem for the Newton-Raphson iteration, which gives the error bound of a given iterate in terms of the two previous iterates. The result may be expressed as

$$|\hat{X}_\infty - \hat{X}_n| \leq |(h_{\max}''/h')|\hat{X}_n - \hat{X}_{n-1}|^2 < \epsilon \tag{38}$$

where the subscript indicates the number of the iteration and $h_{\max}''$ is the maximum of $h''(\hat{X})$ in the iteration interval. It is estimated easily using (35) and (8) as

$$h''_{\max} \approx |h''(\hat{X}_n)| + |h'''(\hat{X}_n)\delta\hat{X}| = |r'| + |(1 + \alpha r)\delta\hat{X}| \tag{39}$$

so that the termination criterion becomes

$$(\delta\hat{X})^2 < \epsilon r[|r'| + |(1 + \alpha r)\delta\hat{X}|]^{-1} \tag{40}$$

This allows termination at at least one cycle sooner than would be anticipated ordinarily and, because of the quadratic nature of the convergence, before roundoff problems arise when working for maximum precision.

Upon completion of the iteration, an accurate value of $\hat{X}$ is available for a final computation of $\hat{U}$, $\hat{C}$, and $\hat{S}$ via (32), r via (14), and the representation coefficients via (23) or (26). The local velocity and position are represented through (24) or (25).

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Free Convection on a Vertical Plate with Concentration Gradients

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Nomenclature

C_p = heat capacity, Btu/lb°R
h = heat-transfer coefficient, Btu/ft²-sec°R
H = enthalpy, Btu/lb
k = thermal conductivity, Btu/ft-sec °R
l = boundary-layer dimension in integral method = δ when $\delta = \delta_T = \delta_i$, ft
L = length of vertical plate, ft
δ = momentum boundary-layer thickness, ft
δ_T = thermal boundary-layer thickness, ft
δ_i = concentration boundary-layer thickness, ft
N = mass flux, lb/ft²-sec
u = boundary-layer velocity, fps
u_1 = velocity outside boundary layer of comparable forced-convective flow, fps
μ = viscosity, lb/ft-sec
θ = $T - \langle T\rangle$, °R
θ_w = $T_w - \langle T\rangle$, °R
τ = shear stress, lb/ft-sec
q = heat flux, Btu/ft²-sec
T = temperature, °R
X_A = mass fraction of component A, dimensionless
ψ = $X_A - \langle X_A\rangle$, dimensionless
ψ_0 = $X_{A0} - \langle X_A\rangle$, dimensionless
ν = μ/ρ = kinematic viscosity, ft²/sec
ρ = density, lb/ft³
β = $-(1/\rho)(\partial\rho/\partial T)_{p,X_A}$
ξ = $-(1/\rho)(\partial\rho/\partial_{X_A})_{p,T}$
$N_{\rm Gr}$ = $g\langle\beta\rangle\,\theta_w z^3/\nu^2$ = Grashof number
$N_{{\rm Gr}_{AB}}$ = $g\langle\xi\rangle\,\psi_0 z^3/\nu^2$ = mass-transfer Grashof number
$N_{{\rm Gr}_L}$ = $g\langle\beta\rangle\,\theta_w L^3/\nu^2$ = average Grashof number
$N_{{\rm Gr}_{AB_L}}$ = $g\langle\xi\rangle\,\psi_0 L^3/\nu^2$ = average mass-transfer Grashof number
$N_{\rm Pr}$ = $C_p\mu/k$ = Prandtl number
$N_{\rm Sc}$ = ν/D_{AB} = Schmidt number

Subscripts

0 = at interface
⟨ ⟩ = average
w = wall

FREE convection in a fluid arises because of instabilities caused by density differences within the fluid. For a single-component or constant-composition fluid, density varies inversely with temperature. In a system of varying composition, however, density is a function of composition, as well as temperature. A variation in composition within the fluid then will either enhance or retard free convection.

With the use of von Karman's integral methods, Eckert and Jackson[1] and Eckert and Drake[2] obtained expressions for the heat-transfer coefficient for a vertical plate in a constant-composition fluid with turbulent and laminar boundary layers, respectively. For turbulent boundary layers, empirical expressions from forced convection were used for wall shear stress, heat flux, velocity, and temperature profiles.

The purpose of this note is to carry out an extension of the analysis by Eckert et al.[1,2] on a vertical plate for the case of a variable concentration in a binary-component fluid. An approximate method of steady-state solution for a situation such as this is that of von Karman and Pohlhausen. In this integral method, an element of fluid is chosen differentially small in one dimension and of finite length in the other (Fig. 1). Finite length l, as a limit on the integral, is chosen as the largest of the three boundary-layer thickness (hydrodynamic, thermal, and concentration) so that, here, free-stream conditions prevail.

Total mass, energy, and z-directed momentum balances may be given as

$$N_{y=l} + \frac{d}{dz}\int_0^l N_z dy = 0 \tag{1}$$

$$q_w = N_{y=l}\langle H\rangle + \frac{d}{dz}\int_0^l N_z H dy \tag{2}$$

and

$$\langle\rho\rangle\frac{d}{dz}\left[\int_0^l u^2 dy\right]dz = g\langle\rho\rangle dz\int_0^l \langle\beta\rangle(T - \langle T\rangle)\,dy + \left[\int_0 \langle\xi\rangle(X_A - \langle X_A\rangle)dy\right] - \tau_w dz \tag{3}$$

where the equation of state is formed by making a double Taylor series of density in temperature and concentration.[3] Thus

$$\rho = \langle\rho\rangle + (\partial\rho/\partial T)_{\langle T\rangle,\langle X_A\rangle}(T - \langle T\rangle) + (\partial\rho/\partial X_A)_{\langle T\rangle,\langle X_A\rangle}(X_A - \langle X_A\rangle) + \ldots$$

To obtain solutions of these equations, functional forms of the dependent variables (u, T, and X_A) must be known or assumed.

Turbulent Free Convection

The forms of the temperature and velocity profiles for turbulent free convection have been determined by experiment and are given by Eckert[1] as

$$u = u_1(y/\delta)^{1/7}[1 - (y/\delta)]^4 \tag{4}$$

and

$$\theta = \theta_w[1 - (y/\delta)^{1/7}] \tag{5}$$

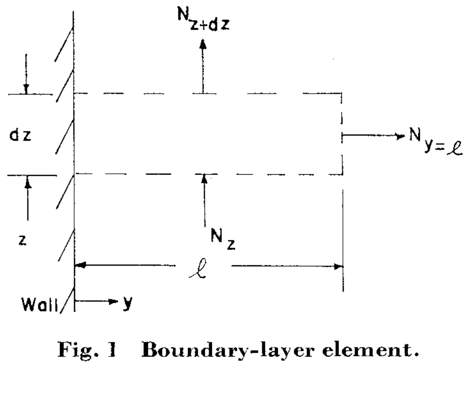

Fig. 1 Boundary-layer element.

Received January 29, 1965; revision received March 31, 1965.

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